Topological and energetic aspects of the random-field Ising model

G. Bertotti,[a] P. Bortolotti, A. Magni, and V. Basso
INRIM, Strada delle Cacce 91, 10135 Torino, Italy

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We show that random-field Ising model spin states are organized in equivalence classes (basins), each class containing the states that can be mutually reached from one another by some field history. Of these basins, only one contains the field-reachable states which can be generated by applying a suitable field history to one of the saturation states. We show that the basins form an oriented graph of which the basin of field-reachable states represents the bottom. The graph exhibits a hierarchical structure which reflects the organization in real space of the spin blocks that can be reversed back and forth by the field in some appropriate field interval without affecting the state of surrounding spins. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2709414]

INTRODUCTION

The random-field Ising model (RFIM) is an extension of the Ising model, with the introduction of frozen disorder at each lattice site. This model provides a convenient framework for the general study of systems characterized by complex energy landscapes. In particular, it has received renewed attention as a tool for the description of hysteresis in disordered magnetic systems. The main reason for that is the fact that the single-spin-flip dynamics induced by the applied field exhibits the so-called return-point memory property,[1] a remarkable property approximately obeyed in many cases by magnetic materials.

Return-point memory is a particular consequence of the more general fact that an ordering relation can be introduced between spin states, ordering that is preserved by the spin-flip evolution under varying field. In this paper we show that the existence of state ordering has, in fact, far-reaching consequences; first of all, the fact that the set of spin states that are stable with respect to single spin-flip can be partitioned into equivalence classes, termed "basins," each class containing the states which can be mutually reached from one another by some field history. These classes clarify and generalize the results at fixed field originally obtained in Refs. 2 and 3. Of these basins, only one contains the "field-reachable" states, which can be generated by applying a suitable field history to one of the saturation states. All the other basins contain states which, although stable, cannot be reached in this way. We will show that the RFIM basins form an oriented graph, of which the basin of field-reachable states represents the bottom. The graph exhibits a hierarchical structure which reflects the organization in real space of the spin blocks that can be reversed back and forth by the field without affecting the state of surrounding spins.

Our results might be helpful in the interpretation not only of the behavior of disordered magnetic systems but also of diffusion processes on random potential surfaces, where a similar hierarchy of energy basins is found and ordering is related to the fact that no return route is possible to previously visited basins.[4]

GRAPH STRUCTURE

We consider the RFIM for N spins $s_i=\pm 1$, $i=1,2,\ldots,N$, on a d-dimensional lattice. The energy of the system is

$$E(\{s_i\};h) = -\frac{J}{2}\sum_i \sum_{\langle j\rangle_i} s_i s_j - \sum_i f_i s_i - h\sum_i s_i, \qquad (1)$$

where $J>0$ is the coupling constant, $\langle j\rangle_i$ is the set of nearest neighbors of site i, and h is the external field. The random fields $\{f_i\}$ are Gaussian distributed, with zero mean and variance σ^2. The local internal field h_i acting on spin s_i is

$$h_i = J\sum_{\langle j\rangle_i} s_j + f_i + h. \qquad (2)$$

The internal field h_i controls the energy change ΔE_i occurring when spin s_i is reversed. In fact, one has $\Delta E_i = 2h_i s_i$, where s_i is the spin value *before* the reversal.

We introduce the notion of field-dependent spin state, defined as the pair $\mathbf{s}=(\{s_i\},h)$, where $\{s_i\}$ represents a certain spin configuration and h is the field applied to that configuration. If $s_i=\mathrm{sign}(h_i)$ for every i, the state is stable with respect to single spin flip because the energy cannot be further reduced by any individual spin reversal. Two stable states of particular importance are the two saturation states $\mathbf{s}_\infty^+=(\{s_i=+1\},h\to+\infty)$ and $\mathbf{s}_\infty^-=(\{s_i=-1\},h\to-\infty)$. To deal with field actions, we introduce the translation operator $\mathcal{T}(\Delta h)$ and the relaxation operator $\mathcal{R}$. The former, defined as $\mathcal{T}(\Delta h)(\{s_i\},h)=(\{s_i\},h+\Delta h)$, describes changes in the external field. Since state stability is a field-dependent property, in general, given the stable state $\mathbf{s}$, the state $\mathbf{s}'=\mathcal{T}(\Delta h)\mathbf{s}$ will not be stable, i.e., there will be unstable spins in it for which $s_i' \neq \mathrm{sign}(h_i')$. The relaxation operator $\mathcal{R}$ transforms an unstable state into a stable one at the same field. Basically, $\mathcal{R}$ is the mathematical representation of what is known in magnetism as a Barkhausen jump. We define the action of $\mathcal{R}$ as that of reversing individual unstable spins in sequence without chang-

[a]Electronic mail: g.bertotti@inrim.it

ing the field until no further unstable spin is left. Since energy is decreased at each reversal step [$\Delta E_i \leqslant 0$ if $s_i \neq \text{sign}(h_i)$], the process must come to an end. The ensuing final state is independent of the order in which unstable spins have been reversed if the state to which $\mathcal{R}$ is applied contains unstable spins with all the same sign.[5] It can be verified that stable states made unstable by the action of $\mathcal{T}(\Delta h)$ fulfill indeed this requirement. Hence the combined evolution operator $\mathcal{U}(\Delta h)=\mathcal{R}\mathcal{T}(\Delta h)$ transforms a given initial stable state at the field h into a uniquely defined final stable state at the field $h+\Delta h$. We will denote the generic sequence $\mathcal{U}(\Delta h_n)\cdots\mathcal{U}(\Delta h_2)\mathcal{U}(\Delta h_1)$ simply by $\mathcal{U}$ and refer to it as the field-history operator. In general, $\mathcal{U}$ will depend on the order in which the various field steps are applied.

Based on $\mathcal{U}$, one can introduce the fundamental notion of mutually reachable or mutually connected states. Given two states $\mathbf{s}$ and $\mathbf{s}'$, we say that they are mutually connected if there exist two field histories $\mathcal{U}$ and $\mathcal{U}'$ such that $\mathbf{s}'=\mathcal{U}\mathbf{s}$ and $\mathbf{s}=\mathcal{U}'\mathbf{s}'$. Mutual connection is an equivalence relation which divides the set of stable states into nonoverlapping equivalence classes B_l, which we call basins. One of these basins, denoted by B_∞, will contain the field-reachable states which can be reached starting from one of the saturation states $\mathbf{s}_\infty^\pm$. The states belonging to B_∞ are the ones involved in usual hysteresis studies under varying field. However, field-reachable states represent, in general, only a small fraction of stable states, the remaining states being organized in a large number of additional basins.

Clearcut insight into the RFIM basin structure is obtained by introducing the notion of state ordering. Given two states $\mathbf{s}=(\{s_i\},h)$ and $\mathbf{s}'=(\{s_i'\},h')$, we say that they are ordered, and we write $\mathbf{s}\leqslant\mathbf{s}'$ if $h\leqslant h'$ and $s_i\leqslant s_i'$ for every i. This definition makes the set of RFIM field-dependent states a partially ordered set.[6] It can be proven that the field-history operator $\mathcal{U}$ is order preserving, i.e., if $\mathbf{s}\leqslant\mathbf{s}'$, then $\mathcal{U}\mathbf{s}\leqslant\mathcal{U}\mathbf{s}'$. Return-point memory is the direct consequence of this property.[1]

State-ordering and order-preserving nature of the $\mathcal{U}$ operator lead to the following central result:[5] Each basin B_l contains a unique pair $(\mathbf{t}_l^-,\mathbf{t}_l^+)$ of ordered states, $\mathbf{t}_l^-\leqslant\mathbf{t}_l^+$, such that $\mathbf{t}_l^-\leqslant\mathbf{s}\leqslant\mathbf{t}_l^+$ for every state $\mathbf{s}\in B_l$. We call $(\mathbf{t}_l^-,\mathbf{t}_l^+)$ the "twin states" associated with B_l. This result has two key consequences.[5] Let $h_l^\pm$ be the fields associated with $\mathbf{t}_l^\pm$, respectively, and h the field associated with $\mathbf{s}$. Then (1) if one applies to $\mathbf{s}\in B_l$ any arbitrary field history spanning field values within the interval (h_l^-,h_l^+), the final state still belongs to B_l; (2) $\mathcal{U}(h_l^+-h)\mathbf{s}=\mathbf{t}_l^+$ and $\mathcal{U}(h_l^--h)\mathbf{s}=\mathbf{t}_l^-$ for every state $\mathbf{s}\in B_l$. Thus, as long as one applies fields within (h_l^-,h_l^+), one remains inside the basin B_l, whereas as soon as one applies fields outside that interval, one leaves the basin through exit states that are the same for every $\mathbf{s}\in B_l$ and precisely coincide with $\mathbf{t}_l^+$ under increasing field and $\mathbf{t}_l^-$ under decreasing field. This means that the effect of arbitrary field histories on RFIM states can be represented by the oriented graph shown in Fig. 1, of which the basins represent the nodes. Each basin is characterized by two outward arrows identifying the basins reached when exiting the basin under increasing or decreasing field. On the other hand, there is no limitation to the number of inward arrows representing ways to reach the basin from other basins. The graph is oriented in the sense that the arrows can form no closed path. In fact, if one such path existed, there would exist mutually connected states belonging to different basins, which is impossible by definition. All the arrow paths eventually reach the basin B_∞, which represents the graph bottom. In fact, one knows that by applying large positive or negative fields to any arbitrary RFIM state, one will eventually reach $\mathbf{s}_\infty^+$ or $\mathbf{s}_\infty^-$, which are just the twin states associated with B_∞. At the top of the graph, there will exist one or more maximal basins having no inward arrows. These basins are, in a sense, the seeds starting from which one can generate the entire basin graph. In general, the basin graph will reflect the statistics of $\{f_i\}$. Quantities like the total number of basins or the number of maximal basins will be drastically different for different values of the ratio σ/J. For example, it was shown in Ref. 2 that the average number of basins visited when going from a randomly chosen basin to B_∞ is a decreasing power-law function of σ/J. In the limit $\sigma/J\to\infty$ of dominant disorder, the entire graph is reduced to the single basin B_∞.

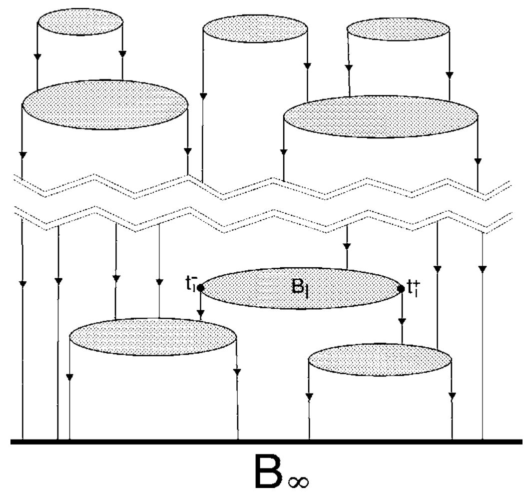

FIG. 1. Schematic representation of top and bottom parts of basin graph. B_∞ is the basin of field-reachable states. The basins at the top with no inward arrows are examples of maximal basins.

HIERARCHICAL STRUCTURE

The graph structure of Fig. 1 is rather abstract and its connection to the spin behavior in real space is not obvious. In order to clarify this aspect, we go back to the notion of twin states. Given a certain pair of twin states $(\mathbf{t}^-,\mathbf{t}^+)$, consider all the states that are ordered with respect to them: $\mathbf{t}^-\leqslant\mathbf{s}\leqslant\mathbf{t}^+$. This set of states is broader than the single basin associated with the twin states and will thus consist of several basins. What all these states have in common is the fact that $\mathcal{U}(h_l^+-h)\mathbf{s}=\mathbf{t}_l^+$ and $\mathcal{U}(h_l^--h)\mathbf{s}=\mathbf{t}_l^-$. In fact, these relations derive from the ordering $\mathbf{t}^-\leqslant\mathbf{s}\leqslant\mathbf{t}^+$ and the properties of the $\mathcal{U}$ operator with no reference to the notion of basin.[5] Thus the states $(\mathbf{t}^-,\mathbf{t}^+)$ can be interpreted as the exit states not only from the basin associated with them, but also from the group of basins satisfying the ordering $\mathbf{t}^-\leqslant\mathbf{s}\leqslant\mathbf{t}^+$. We will call such a group a "container." Therefore, each pair of twin states has a double role: it identifies a specific basin and, at the same time, it identifies a container of several basins. Containers reveal the existence of a hierarchical structure in the basin graph, because the basins belonging to a certain container will be themselves organized in containers and so on, in a nested fashion. At the root of the hierarchy there is the $(\mathbf{s}_\infty^-,\mathbf{s}_\infty^+)$ container, which contains all stable states.

Containers have a direct physical counterpart in the notion of active and inactive spin sites. Consider the twin states $(\mathbf{t}^-,\mathbf{t}^+)$ and let (h^-,h^+) be the corresponding fields. Since these states are ordered, one can divide the N spin sites into two subsets, say $\{s_i\}_I$ and $\{s_i\}_A$, characterized by the fact that the former contains the spins which have the same orientation in both $\mathbf{t}^-$ and $\mathbf{t}^+$, whereas the latter contains the spins which are equal to -1 in state $\mathbf{t}^-$ and to $+1$ in state $\mathbf{t}^+$. Any state belonging to the $(\mathbf{t}^-,\mathbf{t}^+)$ container, i.e., such that $\mathbf{t}^-\leqslant\mathbf{s}\leqslant\mathbf{t}^+$, corresponds to a spin configuration where $\{s_i\}_I$ is the same as for the twin states, whereas the spins in $\{s_i\}_A$ are partly equal to $+1$ and partly to -1. The twin-state nature of $(\mathbf{t}^-,\mathbf{t}^+)$ is revealed by two facts. On the one hand, as long as one applies field histories within the interval (h^-,h^+), one generates states for which $\{s_i\}_I$ remains inactive. One can restrict attention to the active spins only and interpret $(\mathbf{t}^-,\mathbf{t}^+)$ as saturation states. On the other hand, as soon as one applies fields outside (h^-,h^+), some spins belonging to $\{s_i\}_I$ are reversed, after which there exists no field history capable of bringing the system back to $\mathbf{t}^+$ or $\mathbf{t}^-$.

This analysis shows that a container basically represents a group of spins $\{s_i\}_A$ which, by applying fields in some appropriate interval, can be reversed back and forth without affecting the surrounding spins. One can then look for partitions of $\{s_i\}_A$ into active/inactive subgroups, which amounts to identifying containers inside the initial container. This process can be continued until one reaches maximal basins corresponding to elementary spin groups that cannot be further split into active/inactive subgroups. In particular, it may happen that these maximal basins correspond to individual active spins in some narrow field interval where all the remaining spins stay inactive.

It is worth remarking that the spin group $\{s_i\}_A$ associated with a container is itself a random-field Ising system. However, the random-field statistics will be altered because the random-field values involved must be precisely those permitting the container to exist. In addition, there will be additional contributions to the random-field statistics due to the interaction between spins in $\{s_i\}_A$ and $\{s_i\}_I$. These considerations suggest that, in principle, it should be possible to deduce the basic properties of the basin graph and the container hierarchy from the random-field statistics and the dimensionality of the spin lattice.

An important aspect that will be addressed in future work is that of the energy properties of the graph and container structure. In particular, one would like to comprehend where states with specific energy properties, for example, the ground state, tend to be located in the graph. We have shown that the container hierarchy amounts to identifying spin configurations $\{s_i\}_I$ which are particularly stable in some appropriate field interval, a property which is expected to have a direct energetic counterpart.[7]

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